Example 1 The license plates in Texas currently have the form ABC 1234 (i.e. 3 capital letters followed by 4 integers.) How many possible license plates are there?

Solution: $\underline{26}\ \underline{26}\ \underline{26}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10}$ 7 slots

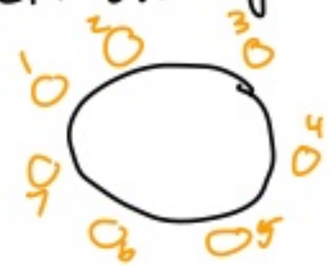
Ans $26^3 \cdot 10^4$ possible license plates.

Example 2 How many palindromic bit strings of length 11 are there? eg 11011111011

Solution $\underset{\text{choices}}{\underline{2}} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{1} \cdot \underline{1} \cdot \underline{1} \cdot \underline{1} \cdot \underline{1}$ 11 slots

$\Rightarrow$ $\boxed{2^6 \text{ palindromic bit strings of length 11}}$

Example 3 How many ways can 7 people sit around a round table, if two seatings are considered the same if each person has the same two neighbors on the left and right?



$\left(\begin{array}{c}\text{\# Arrangements of 7 people in} \\ \text{a line}\end{array}\right) = 7!$

We are over-counting by a factor of $7 \cdot 2$

Total # of seatings $= \dfrac{\cancel{7}\cdot 6\cdot 5\cdot 4\cdot 3\cdot \cancel{2}}{\cancel{7}\cdot\cancel{2}} = \boxed{360 \text{ seatings}}$

$\boxed{\text{Example (4)}}$ How many possible University Soccer Committees could be formed at TCU, if they must have 2 undergraduates, 1 grad student, and one full-time faculty member?

FYI: 11,049 undergrads, 1,889 grads, 709 faculty.

4 slots to fill: $\boxed{\underset{u}{(11{,}049)}\ \underset{u}{(11{,}048)}}\ \underset{g}{(1889)}\ \underset{f}{(709)}$

overcounting by factor of 2

choices

$$\text{\# of possible committees} = \boxed{\frac{(11049)(11048)(1889)(709)}{2}}.$$

$\boxed{\text{Thm}}$ **Principle of Inclusion-Exclusion** (simplest version)

Two tasks/events T_1, T_2 that are not mutually exclusive,

$$\left[\text{the \# of outcomes of } (T_1 \ \boxed{\text{OR}}\ T_2)\right] = \begin{pmatrix}\text{\# of}\\ \text{outcomes}\\ \text{for } T_1\end{pmatrix} + \begin{pmatrix}\text{\# of}\\ \text{outcomes}\\ \text{for } T_2\end{pmatrix} - \begin{pmatrix}\text{\# of}\\ \text{outcomes}\\ T_1 \text{ and } T_2\end{pmatrix}$$

Example (5) How many positive integers ≤ 100 are divisible by 6 or by 9?

Solution: $\begin{pmatrix}\text{\# of positive integers} \leq 100\\ \text{divisible by } 6\end{pmatrix} = \left\lfloor \frac{100}{6} \right\rfloor = 16$

6, 12, 18

$\begin{pmatrix}\text{\# of pos. integers} \leq 100\\ \text{divisible by } 9\end{pmatrix} = \left\lfloor \frac{100}{9} \right\rfloor = 11$

$\begin{pmatrix}\text{\# of pos. integers} \leq 100\\ \text{divisible by both } \underset{2\cdot 3}{6} \,\&\, \underset{3\cdot 3}{9}\\ \text{ie \# divisible by } 18\end{pmatrix} = \left\lfloor \frac{100}{18} \right\rfloor = 5$

Ans: $16 + 11 - 5 = \boxed{22}$.

Example 6 Suppose that p & q are two prime numbers.

Find the number of pos. integers $\leq pq$ that are relatively prime to pq.

Multiples of p : $\underbrace{p, 2p, \cdots qp}_{q \text{ of them}}$.

Multiples of q : $\underbrace{q, 2q, \cdots, pq}_{p \text{ of them}}$.

Multiples of pq : $\underbrace{pq}_{1}$

#s $\leq pq$ are <u>not</u> rel. prime to pq are

$$q + p - 1$$

$\therefore$ # of #s $\leq pq$ rel. prime to pq

$$= pq - (q + p - 1) = \boxed{pq - q - p + 1}$$

Exercise 7 How many subsets are there of a set of N objects?

Solution: N slots $\underset{1}{\underline{2}}\ \underset{2}{\underline{2}}\ \underset{3}{\underline{2}} \cdots \underset{N}{\underline{2}}$

possibilities (in or not in subset.)

$$\Rightarrow \boxed{2^N \text{ subsets}}.$$

Thm Pigeonhole Principle: (balls in boxes)

If we place $k+1$ objects into k boxes, there is at least one box with two items in it.

Example 8 If we choose 27 words at random. By the P.P., at least 2 of the words end on the same letter. [BTW, also at least two of the words start with the same letter.]

Generalized Pigeonhole Principle - If we place N objects into k boxes, then at least one box has at least $\lceil N/k \rceil$ objects in it.

Example 9 Prove that in our class of 33 students, at least 4 people will get exactly the same course grade.

Proof: The possible grades are A, A-, B+, B, B-, C+, C, D, F (9 boxes). By the GPP, one grade ~~has~~ is given to at least $\left\lceil \frac{33}{9} \right\rceil$ students. ■.

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Example 10 Suppose that 7 integers (distinct) are chosen from $\{1, 2, 3, \ldots, 10\}$. Prove that there are at least 2 pairs of integers among those chosen that sum to 11.

$$\left.\begin{array}{l} 1+10 \\ 2+9 \\ 3+8 \\ 4+7 \\ 5+6 \end{array}\right\} \text{5 boxes} \leftarrow \text{7 integers.}$$

G.P.P $\Rightarrow$ at least one box has 2 integers in it.

Since each box contains only 2 integers, at least two of the boxes must have both integers chosen. ∅